

Technological Inefficiency and the Skewness of the Error Component in Stochastic Frontier Analysis

Citation for published version (APA):

Carree, M. A. (2002). Technological Inefficiency and the Skewness of the Error Component in Stochastic Frontier Analysis. *Economics Letters*, 77, 101-107. [https://doi.org/10.1016/S0165-1765\(02\)00119-2](https://doi.org/10.1016/S0165-1765(02)00119-2)

Document status and date:

Published: 01/01/2002

DOI:

[10.1016/S0165-1765\(02\)00119-2](https://doi.org/10.1016/S0165-1765(02)00119-2)

Document Version:

Publisher's PDF, also known as Version of record

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.umlib.nl/taverne-license

Take down policy

If you believe that this document breaches copyright please contact us at:

repository@maastrichtuniversity.nl

providing details and we will investigate your claim.

Download date: 04 Dec. 2019

Technological inefficiency and the skewness of the error component in stochastic frontier analysis

Martin A. Carree^{a,b,*}

^a*Erasmus University Rotterdam, Rotterdam, The Netherlands*

^b*Faculty of Economics and Business Administration, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands*

Received 30 August 2001; accepted 20 March 2002

Abstract

This paper concentrates on negatively skewed one-sided distributions as an explanation of the occurrence of positive (negative) skewness in the case of stochastic production (cost) frontier analysis. It takes as an example the binomial distribution that can have negative or positive skew and derives the method-of-moments estimators. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Production frontier; Cost frontier; Skewness; Binomial distribution

JEL classification: C13; D24

1. Introduction

A popular econometric technique to estimate the extent of firm inefficiency is stochastic frontier analysis. A pioneering publication on the econometric estimation of stochastic frontiers is Aigner et al. (1977). They suggest an estimation procedure in which a production frontier is estimated along with a two-part composed error term. The first part of the error term consists of conventional statistical noise and is usually assumed to be normally distributed. The second part represents firm inefficiency and is assumed to follow a one-sided distribution. Several distributions have been proposed for the one-sided distribution including the half-normal distribution, the truncated normal distribution, the exponential distribution and the two-parameter gamma-distribution (Greene, 1990). Each of these one-sided distributions has a positive skewness. Li (1996) considers the case of the symmetric one-sided uniform distribution.

*Tel.: +31-43-388-3763; fax: +31-43-388-4877.

E-mail address: m.carree@mw.unimaas.nl (M.A. Carree).

A common problem in the use of the stochastic production frontier analysis is that the estimated skewness of the residuals is positive. Green and Mayes (1991) report that for a sample of 151 UK industries, 32% showed a positive skewness of the combined residual and that for a sample of 140 Australian industries a similar problem was encountered in 35% of the cases.¹ A positive skewness is considered problematic because it cannot be reconciled with a one-sided distribution of inefficiencies that is positively skewed. Green and Mayes argue that, apart from possible misspecification of the production functions, this either indicates ‘super efficiency’ (all firms in the industry are efficient) or the inappropriateness of the technique of frontier production function analysis to measure inefficiencies. They overlook one important additional possibility: that of negatively skewed one-sided distributions of inefficiencies. In this paper we consider this possibility and show that it has important consequences for the interpretation of the skewness of the error term as a measure of technological inefficiency.

2. The model of stochastic production frontier analysis

We consider the following production frontier model for a sample of N firms:²

$$y_i = \alpha + x_i' \beta + \varepsilon_i \quad i = 1, \dots, N \quad (1)$$

with $\varepsilon_i = v_i - u_i$ being the composite error term. The commonly made assumption for the statistical noise term v_i is that it is i.i.d. $N(0, \sigma^2)$. The $u_i \geq 0$ represents the technological inefficiency of firm i . The two parts of the error term are assumed to be independently distributed. The y_i and the x_i -vector stand for the output and the inputs used in the production process, respectively. The composite error term ε_i has an expected value equal to $E\varepsilon_i = -Eu_i$ and a third central moment equal to:

$$E(\varepsilon_i - E\varepsilon_i)^3 = E(v_i - u_i + Eu_i)^3 = -E(u_i - Eu_i)^3. \quad (2)$$

Therefore, a positively skewed distribution of the inefficiencies u_i implies that the adjusted error term $\varepsilon_i - E\varepsilon_i$ has a negative skewness. Now it has been common practice to use a positively skewed one-sided distribution. In fact, in case one wants the one-sided error u_i to have an *unbounded* range, then most well-known distributions are in fact positively skewed. Examples include the gamma distribution (including the exponential distribution), the Poisson distribution, the negative binomial distribution, the truncated normal distribution and the half-normal distribution. But there is at least one well-known distribution defined on $[0, \infty)$ that may show negative skewness: the Weibull distribution.³ In case one allows for the one-sided error to have a *bounded* range, a longer list of

¹Another example is Mester (1997) who applies the stochastic *cost* frontier analysis and finds that out of 12 US bank districts three have negatively skewed residuals. She carefully remarks that her ‘frontier model with normal–half-normal error term does not fit the data in these districts’ (p. 238).

²Although we concentrate upon the production frontier case, the arguments are similar for the cost frontier model in which the costs of firm i are determined by the cost frontier and an error term of the form $\varepsilon_i = u_i + v_i$ with the one-sided error term $u_i \geq 0$ capturing cost inefficiencies.

³Johnson et al. (1994, p. 633) show that a Weibull distribution has a positive skewness for parameter values up to 3.602 and a negative coefficient of skewness for higher parameter values. Li (1996, p. 222) does not recognize this possibility and argues that a one-sided error component with unbounded range always has a positive skewness.

well-known distributions with (possible) negative skewness becomes available. In the current paper we will examine the binomial $b(n, p)$ -distribution. There is no particular reason to choose this distribution apart from that it allows for both positive and negative skewness.

A simple method-of-moments (MM) estimator for the binomial one-sided distribution can be derived by using the corrected OLS approach (see e.g. Greene, 1990; Olson et al., 1980). This approach implies that firstly the parameters of the production function (1) are estimated using least squares and that secondly the estimated residuals are used to estimate the parameters of the distributions of u_i and v_i . The corrected OLS procedure leads to consistent estimators of the parameters of the production function and of the composed error term distribution.⁴ First define $r_i = u_i - Eu_i$ so that $Er_i = 0$. Because r_i and v_i are independent, we have that $Er_i^j v_i^k = Er_i^j Ev_i^k$ with j and k positive integers. In addition, because v_i is distributed symmetrically we have $Ev_i^k = 0$ if k is an odd positive integer. From the error decomposition $\varepsilon_i - E\varepsilon_i = v_i - r_i$ we find:

$$E(\varepsilon_i - E\varepsilon_i)^2 = Ev_i^2 + Er_i^2 \quad (3)$$

$$E(\varepsilon_i - E\varepsilon_i)^3 = -Er_i^3 \quad (4)$$

$$E(\varepsilon_i - E\varepsilon_i)^4 = Ev_i^4 + 6Ev_i^2 Er_i^2 + Er_i^4. \quad (5)$$

For a normally distributed v_i we insert $Ev_i^2 = \sigma^2$ and $Ev_i^4 = 3\sigma^4$. By combining Eqs. (3) and (5) we have:

$$E(\varepsilon_i - E\varepsilon_i)^4 - 3(E(\varepsilon_i - E\varepsilon_i)^2)^2 = Er_i^4 - 3(Er_i^2)^2. \quad (6)$$

From Eqs. (4) and (6) an MM-estimator for a two-parameter distribution can be derived in analogue to Greene (1990). For the binomial distribution we have that $Er_i^2 = np(1-p)$, $Er_i^3 = np(1-p)(1-2p)$ and $Er_i^4 = 3(np(1-p))^2 + np(1-p)(1-6p+6p^2)$.⁵ From the third central moment it is obvious that the binomial distribution has a positive skewness for p between zero and one half and a negative skewness for p between one half and unity. After replacing the k th central moments of ε_i with the sample analogues $\hat{\mu}_k = \sum_i \varepsilon_i^k / N$ we have the following two equations that determine MM-estimates for n and p :

$$\hat{\mu}_3 = -np(1-p)(1-2p) \quad \hat{\mu}_4 - 3\hat{\mu}_2^2 = np(1-p)(1-6p+6p^2). \quad (7)$$

That is, the values of p determine the signs of the sample moments $\hat{\mu}_3$ (skewness) and $\hat{\mu}_4 - 3\hat{\mu}_2^2$ (kurtosis adjusted for the value for normality). Assume that $\hat{\mu}_3 \neq 0$ and define $x = (\hat{\mu}_4 - 3\hat{\mu}_2^2) / \hat{\mu}_3$.

⁴The method-of-moment estimators for the two parameters of the binomial distribution are derived using the second, third and fourth sample moments. The use of higher-order moments makes the estimators vulnerable to outliers and may lead to poor small sample properties. Greene's (1990) MM-estimators for the two-parameter Gamma-distribution suffer from this problem as well. Hosking (1990) proposes to use L-moments instead of the standard measures of skewness and kurtosis to achieve relatively small sensitivity to outliers. However, the derivation of estimators based upon L-moments is beyond the scope of the present paper.

⁵See for example Johnson et al. (1992, p. 107).

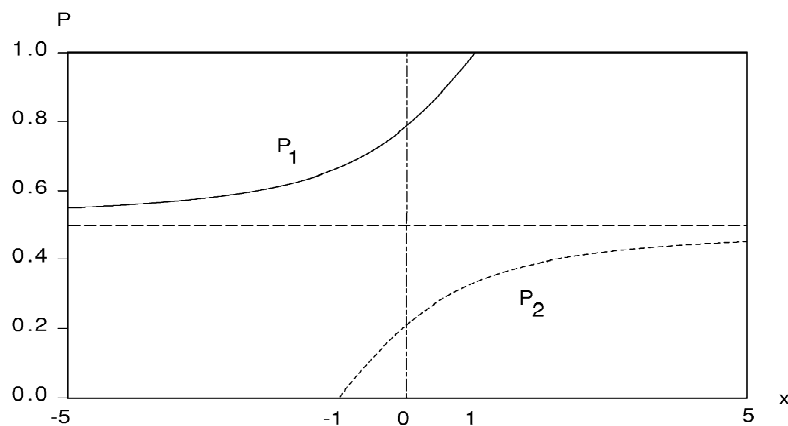


Fig. 1. Method-of-moments estimator p as a function of $x = (\hat{\mu}_4 - 3\hat{\mu}_2^2)/\hat{\mu}_3$.

According to Eq. (7) this should be equal to $(6p^2 - 6p + 1)/(2p - 1)$. From this we derive the two possible values of p as a function of x :

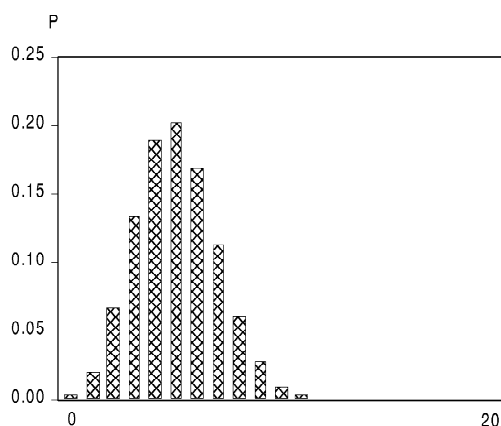
$$p_1 = \frac{1}{2} + \frac{1}{6}x + \frac{1}{6}\sqrt{x^2 + 3} \quad p_2 = \frac{1}{2} + \frac{1}{6}x - \frac{1}{6}\sqrt{x^2 + 3}. \quad (8)$$

For values of x less than -1 only the p_1 -solution is allowed. For values of x in excess of $+1$ only the p_2 -solution is allowed. For values of x in between -1 and $+1$ the signs of $\hat{\mu}_3$ and $\hat{\mu}_4 - 3\hat{\mu}_2^2$ determine which of the two solutions is appropriate. That is, if skewness is positive ($\hat{\mu}_3 > 0$) then the p_1 -solution will be chosen otherwise the p_2 -solution. In Fig. 1 the graphs of p_1 and p_2 as a function of x are given.

Not all combinations of the empirical values for $\hat{\mu}_3$ and $\hat{\mu}_4 - 3\hat{\mu}_2^2$ allow for MM-estimates. In fact, in case $\hat{\mu}_4 - 3\hat{\mu}_2^2 > \hat{\mu}_3 > 0$ or $\hat{\mu}_4 - 3\hat{\mu}_2^2 > -\hat{\mu}_3 > 0$ there are no valid MM-estimates for p . To derive the MM-estimator for $Ev_i^2 = \sigma^2$ using Eq. (3) we also require that $\hat{\mu}_2$ should not be less than $np(1-p)$ after inserting the MM-estimates of p and n . It is a question of empirics whether these violations, which would indicate the implausibility of the one-sided distribution to be of a binomial type, are encountered.

3. What do negative and positive skewnesses actually measure?

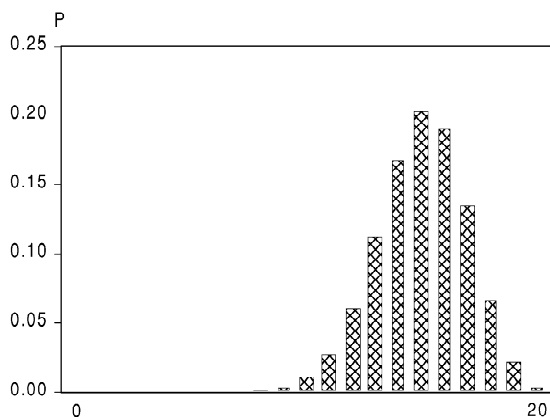
Empirical studies using the production frontier approach have been assuming positively skewed one-sided distributions (and, hence, negatively skewed adjusted composite error terms). As a consequence, when a positive value of $\hat{\mu}_3$ was found, the only logical conclusion could be that there had been unfortunate sampling from a distribution that had in fact a *population* skewness below zero. As Monte Carlo studies have shown, this is a possibility that may occur relatively frequently in case the one-sided distribution has a small variance in comparison with the symmetric error distribution (see e.g. Fan et al., 1996; Green and Mayes, 1991). Waldman (1982) showed that resorting to a maximum likelihood procedure instead of a corrected OLS procedure does not resolve the problem. In

Fig. 2. Probabilities of $b(20, 0.25)$.

fact, he has shown that, in case of a positive $\hat{\mu}_3$, the ML estimator for the stochastic frontier model is simply OLS for the slope vector and the absence of any efficiencies (variance of u_i is zero). When an industry showed positive skewness of the residuals it was therefore assumed that there were little if any inefficiencies. Green and Mayes (1991) argue that a ‘positive skew implies that establishments in the industry are ‘super efficient’, rather than inefficient’ (p. 528).

In contrast to the conclusion of ‘super efficiency’ in case of a positive skewness, the example of the binomial distribution shows that a positive skewness suggests a one-sided distribution that has low probabilities for small inefficiencies and high probabilities of large inefficiencies. For the binomial distribution it indicates that p is between one half and one. Hence, only a small fraction of the firms or plants attain a level of productivity close to the frontier while a large fraction attains considerable inefficiencies. See Fig. 3 in which we have n equal to 20 (inefficiency categories) and p equal to 0.75. The case of a negative skewness implies that only a small fraction of firms are lagging behind. See Fig. 2 in which we have p equal to 0.25.

Figs. 2 and 3 can also be interpreted as two stages in an industry characterized by the *cycle of*

Fig. 3. Probabilities of $b(20, 0.75)$.

innovation and imitation. Assume that the productivities in an industry are characterized by Fig. 2. In case one firm achieves an important innovation by which it can increase productivity, it becomes dominant (in terms of productivity), and Fig. 3 may emerge. Other firms will then seek to imitate the successful firm and Fig. 2 may be restored. This process of ‘transient dominance’ in an industry would lead to a cyclical time series pattern of positive and negative skewness of residuals of the stochastic production frontier analysis: innovation leads to positive skew, imitation leads again to negative skew.

What is the more likely interpretation of a positive skewness of the composite error term in stochastic production frontier analysis? On the one hand, it may be an unfortunate draw and the industry may be characterized by ‘super efficiency’ (or at least, the symmetric error term v_i dominates the one-sided distributed u_i). On the other hand, the large majority of firms may be quite inefficient (like in Fig. 3). The two interpretations are completely different, either indicating no efficiencies or large inefficiencies. An argument against the first interpretation is that relative productivities of plants are persistent over time (e.g. Bailey et al., 1992). In case there would have been no inefficiencies (i.e. the error term is determined completely by statistical noise $\varepsilon_i = v_i$) one would not expect such persistence, unless the statistical noise has strong autocorrelation.

4. Conclusion

An important methodological problem in stochastic frontier analysis has been the occurrence of residuals being skewed in the ‘wrong’ direction. In the case of production frontiers, many times positively skewed residuals have been found, while in the case of cost frontiers, negative skewnesses have been quite common. The traditional solution to the problem has been to argue that there are no inefficiencies and to put the variance of the one-sided distribution equal to zero. This solution fails to be convincing. This paper suggests a different solution: the one-sided distribution of inefficiencies may be negatively skewed (in case of production frontiers) or positively skewed (in case of cost frontiers). This does not imply that the traditional solution arguing for unfortunate sampling is impossible, but that a better approach to the stochastic frontier analysis, in which a comparison is made of several industries (or regions or time periods), is to use a distribution allowing for positive and negative skewness.

Acknowledgements

The author is grateful to the Royal Netherlands Academy of Arts and Sciences (KNAW) for financial support and to Peter Brouwer for our discussions on the application of stochastic production frontier models to Dutch construction industry data.

References

- Aigner, D.J., Lovell, C.A.K., Schmidt, P., 1977. Formulation and estimation of stochastic frontier production function models. *Journal of Econometrics* 6, 21–37.

- Bailey, M.N., Hulten, C., Campbell, D., 1992. Productivity dynamics in manufacturing plants. *Brookings Papers on Economic Activity: Microeconomics*, 187–267.
- Fan, Y., Li, Q., Weersink, A., 1996. Semiparametric estimation of stochastic production frontier models. *Journal of Business and Economics Statistics* 14, 460–468.
- Green, A., Mayes, D., 1991. Technical inefficiency in manufacturing industries. *Economic Journal* 101, 523–538.
- Greene, W.H., 1990. A Gamma-distributed stochastic frontier model. *Journal of Econometrics* 46, 141–164.
- Hosking, J.R.M., 1990. L-moments: analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society, Series B* 52, 105–124.
- Johnson, N.L., Kotz, S., Balakrishnan, N., 1994. In: 2nd Edition. *Continuous Univariate Distributions*, Vol. 1. John Wiley, New York.
- Johnson, N.L., Kotz, S., Kemp, A.W., 1992. In: 2nd Edition. *Univariate Discrete Distributions*. John Wiley, New York.
- Li, Q., 1996. Estimating a stochastic production frontier when the adjusted error is symmetric. *Economics Letters* 52, 221–228.
- Mester, L.J., 1997. Measuring efficiency at US banks: accounting for heterogeneity is important. *European Journal of Operational Research* 98, 230–242.
- Olson, J.A., Schmidt, P., Waldman, D.M., 1980. A Monte Carlo study of estimators of the stochastic frontier production function. *Journal of Econometrics* 13, 67–82.
- Waldman, D.M., 1982. A stationary point for the stochastic frontier likelihood. *Journal of Econometrics* 18, 275–279.